

CERN-TH/97-175

CU-TP-852

hep-th/9707251

# Four-dimensional BPS-spectra via $M$ -theory

Måns Henningson

*Theory Division, CERN**CH-1211 Geneva 23, Switzerland*`henning@nxth04.cern.ch`

Piljin Yi

*Physics Department, Columbia University**New York, NY 10027, USA*`piljin@phys.columbia.edu`

## Abstract

We consider the realization of four-dimensional theories with  $N = 2$  supersymmetry as  $M$ -theory configurations including a five-brane. Our emphasis is on the spectrum of massive states, that are realized as two-branes ending on the five-brane. We start with a determination of the supersymmetries that are left unbroken by the background metric and five-brane. We then show how the central charge of the  $N = 2$  algebra arises from the central charge associated with the  $M$ -theory two-brane. This determines the condition for a two-brane configuration to be BPS-saturated in the four-dimensional sense. By imposing certain conditions on the moduli, we can give concrete examples of such two-branes. This leads us to conjecture that vectormultiplet and hypermultiplet BPS-saturated states correspond to two-branes with the topology of a cylinder and a disc respectively. We also discuss the phenomenon of marginal stability of BPS-saturated states.

CERN-TH/97-175

July 1997

## 1. Introduction

Many interesting results about supersymmetric theories have been obtained by considering configurations of branes in higher dimensional theories. In particular, Witten [1] has recently shown how four-dimensional theories with  $N = 2$  extended supersymmetry can be realized as  $M$ -theory configurations including a supersymmetric five-brane. In many cases, this construction gives an easy way to determine the spectral curve and the associated meromorphic one-form that appears in the Seiberg-Witten formulation [2] of  $N = 2$  supersymmetric gauge theories.

Our focus in this paper is on the spectrum of massive BPS-saturated states in such theories. These states enjoy a particular stability property, that ensures that they can only decay as certain so called marginal stability domain walls of codimension one in the moduli space of vacua are crossed. Given the existence of a BPS-saturated state with certain quantum numbers, Seiberg-Witten theory gives a formula for its contribution to the central charge of the  $N = 2$  algebra and thus for its mass. However, it is in general a difficult problem to determine the spectrum of such states that really exist at a given point in the moduli space. (See for example [3] for the case of the pure  $N = 2$   $SU(2)$  Yang-Mills theory.) The  $M$ -theory interpretation has the conceptual advantage of giving a fairly concrete picture of such states: They correspond to two-branes with boundary on the five-brane. The homology class of the boundary determines the quantum numbers of the state, and BPS-saturation amounts to the world-volume being minimal in an appropriate sense. The problem of finding the BPS-saturated spectrum is therefore in principle reduced to the problem of finding such minimal world-volumes. (Another approach, that amounts to finding geodesic curves on the five-brane, was initiated in [4].)

This paper is organized as follows: In section two, we determine the unbroken supersymmetries in these  $M$ -theory configurations. The eleven-dimensional manifold on which the theory is defined must contain a four-dimensional hyper Kähler manifold as a factor. The five-brane configuration is described by a two-dimensional surface in this space that is holomorphically embedded with respect to one of its complex structures. In section three, we show how the eleven-dimensional central charge associated with a two-brane gives rise to the central charge of the  $N = 2$  algebra in four dimensions, and determine the conditions for a two-brane to be BPS-saturated. It turns out that the two-brane must also be described by a two-dimensional surface which is holomorphically embedded, but with respect to another complex structure which is orthogonal to the original one. The

remaining freedom in choosing this second complex structure corresponds to the phase of the central charge of the BPS-saturated state. In section four, we give some concrete examples of BPS-saturated states by imposing certain reality conditions on the moduli. These examples lead to the conjecture that BPS-saturated vectormultiplets and hypermultiplets correspond to surfaces with the topology of a cylinder and a disc respectively. We also discuss the phenomenon of decay of BPS-saturated states across domain walls of marginal stability and its relation to the mutual non-locality of the constituent states.

Throughout this paper, we exemplify the general discussion by considering the case of the low-energy effective theory of  $N = 2$  QCD, i.e.  $SU(N_c)$  super Yang-Mills theory with some number  $N_f$  of hypermultiplet quarks in the fundamental representation.

While preparing this manuscript, we were informed that closely related issues would be addressed in [5].

## 2. $N = 2$ supersymmetry in $d = 4$ dimensions from $M$ -theory

Following [1], we consider  $M$ -theory on an eleven-dimensional manifold  $M^{1,10}$  of the form

$$M^{1,10} \simeq \mathbf{R}^{1,3} \times X^7, \quad (2.1)$$

where the first factor is the four-dimensional Minkowski space-time and the second factor is some (non-compact) seven-manifold. Furthermore, we introduce a five-brane, the world-volume of which fills  $\mathbf{R}^{1,3}$  and defines a two-dimensional surface  $\Sigma$  in  $X^7$ .

The supercharges  $Q_A$ ,  $A = 1, \dots, 32$  of  $M$ -theory transform as a Majorana spinor under the eleven-dimensional Lorentz group. In a flat space background they fulfill the supertranslations algebra [6]

$$\{Q_A, \bar{Q}_B\} = (\Gamma^M)_{AB} P_M + (\Gamma^{MN})_{AB} Z_{MN} + (\Gamma^{MNPQR})_{AB} W_{MNPQR}. \quad (2.2)$$

Here  $P_M$  is the eleven-dimensional energy-momentum ( $M, N, \dots = 0, 1, \dots, 10$ ), and the central charges  $Z_{MN}$  and  $W_{MNPQR}$  are associated with the two-brane and the five-brane respectively:

$$\begin{aligned} Z^{MN} &\sim \int_{2\text{-brane}} dX^M \wedge dX^N \\ W^{MNPQR} &\sim \int_{5\text{-brane}} dX^M \wedge \dots \wedge dX^R. \end{aligned} \quad (2.3)$$

Part of the eleven-dimensional supersymmetry generated by the  $Q_A$  is spontaneously broken by the background configuration: An unbroken supersymmetry generator must first of all be covariantly constant with respect to the background metric on  $M^{1,10}$ . Furthermore, it follows from the algebra (2.2) and the expression (2.3) for the central charges together with the fact that the mass of a brane is proportional to its area, that an unbroken supersymmetry generator must be of positive chirality with respect to the tangent-space of each five-brane world-volume element. In the rest of this section, we will analyze the implications of these requirements in more detail.

### 2.1. The background metric

We are interested in background configurations with an unbroken  $N = 2$  extended supersymmetry in  $\mathbf{R}^{1,3}$ . To this end, we first require the holonomy group of  $X^7$  to be isomorphic to  $SU(2)$  (rather than to  $SO(7)$  as it would be for a generic seven-manifold). The spinor representation of  $SO(7)$  then contains four singlets with respect to the  $SU(2)$  holonomy group, so  $X^7$  admits four covariantly constant spinors. The holonomy of the background metric on  $X^7$  thus leaves an  $N = 4$  extended supersymmetry in  $\mathbf{R}^{1,3}$  unbroken. Furthermore, the vector representation of  $SO(7)$  contains three singlets with respect to the  $SU(2)$  holonomy group. This means that  $X^7$  must be of the form

$$X^7 \simeq \mathbf{R}^3 \times Q^4, \quad (2.4)$$

where  $\mathbf{R}^3$  is generated by the flows of the three covariantly constant vector fields on  $X^7$ , and  $Q^4$  is a four-manifold of  $SU(2)$  holonomy, i.e. a hyper-Kähler manifold. The 4 covariantly constant spinors of  $X^7$  transform as a pair of doublets under the  $SO(3) \simeq SU(2)/\mathbf{Z}_2$  Lorentz group of  $\mathbf{R}^3$ . Each doublet corresponds to one of the two covariantly constant Majorana spinors on  $Q^4$ , which we will denote by  $\zeta_1$  and  $\zeta_2$ .

Hyper-Kähler geometry can be thought of as a Kähler geometry which admits a family of inequivalent complex structures parametrized by a two-sphere  $S^2$ . In four dimensions, the hyper-Kähler condition is in fact equivalent to the Calabi-Yau condition, i.e.  $Q^4$  is a Ricci-flat Kähler manifold and should therefore admit a covariantly constant holomorphic two-form  $\Omega$ . Given a choice of complex structure  $J$ , i.e. a point on the two-sphere  $S^2$ ,  $\Omega$  is uniquely determined (up to a complex constant) as  $K' + iK''$ , where  $K'$  and  $K''$  are the Kähler forms corresponding to two other complex structures  $J'$  and  $J''$  that are orthogonal to each other and also to the chosen complex structure  $J$ .

## 2.2. The five-brane geometry

Given the background metric described in the previous subsection, we will now determine the constraints on the embedding of a five-brane for the resulting configuration to leave  $N = 2$  supersymmetry unbroken in  $\mathbf{R}^{1,3}$ . First of all, unbroken Poincaré invariance requires that the five-brane world-volume fills all of  $\mathbf{R}^{1,3}$  and defines a two-dimensional surface  $\Sigma$  in  $X^7$ . Furthermore, we will later identify (the double cover of) the Lorentz group  $SO(3)$  of the first factor of (2.4) with the  $R$ -symmetry of the  $N = 2$  algebra. To leave this unbroken,  $\Sigma$  must in fact lie at a single point in  $\mathbf{R}^3$  and define a surface in  $Q^4$ , that by a slight abuse of notation we also call  $\Sigma$ .

A generic five-brane configuration, described by a surface  $\Sigma$  in  $Q^4$ , will break all supersymmetries. To have  $N = 2$  supersymmetry unbroken, we must require the spatial volume element of the surface  $\Sigma$  to be minimized so that it saturates a topological bound. Given a choice of complex structure  $J$  on  $Q^4$ , a useful identity in this regard involves the pull-backs  $K_\Sigma$  and  $\Omega_\Sigma$  of the corresponding Kähler form  $K$  and the holomorphic two-form  $\Omega$  to  $\Sigma$ . A straightforward computation shows that

$$\frac{1}{4} \left( (*K_\Sigma)^2 + |*\Omega_\Sigma|^2 \right) = 1, \quad (2.5)$$

where  $*$  denotes the Hodge-dual with respect to the induced metric on  $\Sigma$ . It follows that the area  $A_\Sigma$  of  $\Sigma$  fulfills

$$2A_\Sigma = 2 \int_\Sigma V_\Sigma = \int_\Sigma V_\Sigma \sqrt{(*K_\Sigma)^2 + |*\Omega_\Sigma|^2} \geq \int_\Sigma V_\Sigma *K_\Sigma = \int_\Sigma K_\Sigma, \quad (2.6)$$

where  $V_\Sigma$  denotes the volume-form of  $\Sigma$  and the inequality is saturated if and only if  $\Omega_\Sigma$  vanishes identically. This is equivalent to requiring  $\Sigma$  to be holomorphically embedded in  $Q^4$  with respect to the complex structure  $J$ . The surface  $\Sigma$  is thus the locus of the equation

$$F = 0 \quad (2.7)$$

for some holomorphic function  $F$  on  $Q^4$ . We emphasise that the choice of the complex structure  $J$  is arbitrary.

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<sup>†</sup> When we are dealing with an infinite surface  $\Sigma$ , the lower bound for the total area does not make much sense. One should regard it as a local statement in that the minimization is with respect to all possible localized perturbation of the surface.

We will now determine the amount of supersymmetry that is left unbroken by this configuration. Recall that the unbroken supersymmetry generators should have positive chirality with respect to each five-brane world-volume element. Since the five-brane world-volume is of the form  $\mathbf{R}^{1,3} \times \Sigma$ , we see that this requirement correlates space-time chirality and chirality with respect to each  $\Sigma$  area element and thus breaks at least half the supersymmetry. In fact, there exist certain linear combinations  $\zeta^+$  and  $\zeta^-$  of the covariantly constant Majorana spinors  $\zeta_1$  and  $\zeta_2$  on  $Q^4$  with *constant* coefficients that have definite and opposite chirality with respect to every area element of  $\Sigma$ , regardless of the form of the holomorphic function  $F$ . The five-brane thus breaks exactly half of the  $N = 4$  supersymmetry, and we are left with  $N = 2$  supersymmetry in  $\mathbf{R}^{1,3}$ .

To verify the existence of such  $\zeta^+$  and  $\zeta^-$ , we consider the spinor representation  $4_s = (2, 1) \oplus (1, 2)$  of the rotation group  $SO(4) \simeq SU(2)_L \times SU(2)_R$  of  $Q^4$ . This representation decomposes into  $2_0 \oplus 1_{-1/2} \oplus 1_{+1/2}$  under the subgroup  $U(2) \simeq SU(2)_L \times U(1)_R$ , where the  $SU(2)_L$  factor is the holonomy group of  $Q^4$  and  $U(1)_R$  is the group of rotations in the tangent space of the holomorphically embedded surface  $\Sigma$ . The two spinors  $\zeta^\pm$  above correspond to the two singlets of  $SU(2)_L$  and can thus be chosen to be chiral eigenstates of opposite sign with respect to an area-element of  $\Sigma$ . But the form of  $\zeta^\pm$  depends only on the choice of  $U(1)_R$  in  $SU(2)_R$ , tantamount to the choice of the complex structure, so  $\zeta^\pm$  are indeed constant linear combinations of  $\zeta_1$  and  $\zeta_2$  as claimed.

### 2.3. The case of $N = 2$ QCD

We will exemplify the general discussion above with the case of the low-energy effective theory of  $SU(N_c)$  super Yang-Mills theory with some number  $N_f$  of quark hypermultiplets in the fundamental representation. As usual, we will impose the restriction that  $0 \leq N_f < 2N_c$  so that the theory is asymptotically free.

In this case,  $Q^4$  is  $\mathbf{R}^3 \times S^1$  with the standard flat metric [1]. We introduce coordinates  $X^4, X^5, X^6$  and  $X^{10}$  on  $Q^4$ , where  $X^{10}$  is periodic with period  $2\pi R$  for some  $R$ . In the following we will set  $R = 1$ . We choose a particular complex structure  $J$  such that  $s = X^6 + iX^{10}$  and  $v = X^4 + iX^5$  are holomorphic coordinates. The corresponding Kähler form is given by

$$K = i(ds \wedge d\bar{s} + dv \wedge d\bar{v}), \quad (2.8)$$

and the covariantly constant holomorphic two-form is

$$\Omega = ds \wedge dv. \quad (2.9)$$

It is convenient to replace  $s$  by the single-valued coordinate  $t = \exp(-s)$ .

Such a  $Q^4$  obviously has a trivial holonomy group and admits two additionally constant Majorana spinors  $\zeta_3$  and  $\zeta_4$ . The background metric on  $Q^4$  thus leaves  $N = 8$  supersymmetry unbroken in  $\mathbf{R}^{1,3}$  rather than  $N = 4$ , but the five-brane again breaks the symmetry down to  $N = 2$ . The reason is that although it is possible to form linear combinations  $\tilde{\zeta}^+$  and  $\tilde{\zeta}^-$  of  $\zeta_3$  and  $\zeta_4$  of definite and opposite chirality with respect to a given area element of  $\Sigma$ , the coefficients are no longer constant but will vary over  $\Sigma$ . The spinors  $\tilde{\zeta}^+$  and  $\tilde{\zeta}^-$  are thus not covariantly constant and do not give rise to unbroken supersymmetries in  $\mathbf{R}^{1,3}$ .

The Riemann surface  $\Sigma$  that describes the five-brane world-volume is given by the equation

$$F(t, v) \equiv t^2 + B(v)t + C(v) = 0, \quad (2.10)$$

where

$$\begin{aligned} B(v) &= \prod_{a=1}^{N_c} (v - \phi_a) \\ C(v) &= \Lambda^{2N_c - N_f} \prod_{i=1}^{N_f} (v - m_i). \end{aligned} \quad (2.11)$$

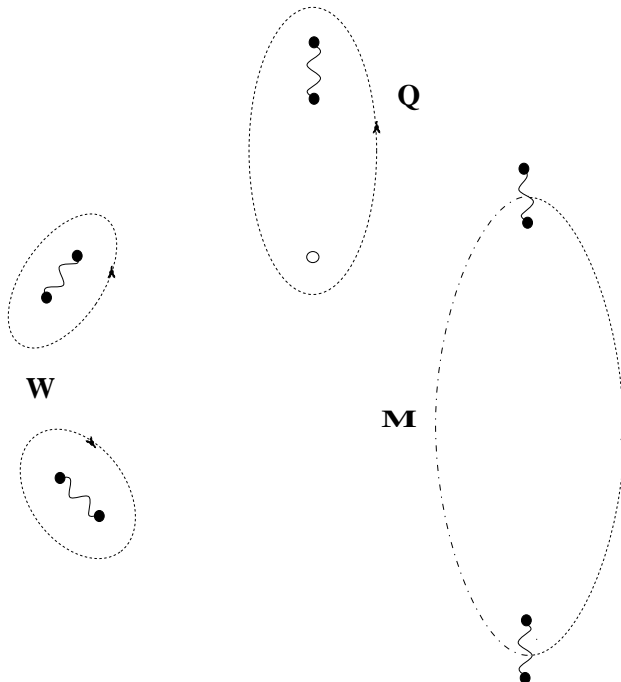
Here the  $m_i$ ,  $i = 1, \dots, N_f$  are the bare masses of the hypermultiplet quarks, the  $\phi_a$ ,  $a = 1, \dots, N_c$  parametrize the moduli space of vacua subject to the constraint

$$\sum_{a=1}^{N_c} \phi_a = 0, \quad (2.12)$$

and  $\Lambda$  is the dynamically generated scale of the theory. This form of  $\Sigma$  was first proposed in [7][8]. If we parametrize the surface  $F = 0$  by  $v$ , we necessarily encounter singular points because  $v$  spans a complex plane or a  $CP^1$ , while the actual surface has genus  $N_c - 1$ . The solutions to the equation  $F = 0$  for a given  $v$ , i.e.

$$t = t_{\pm}(v) = -\frac{B(v)}{2} \pm \sqrt{\left(\frac{B(v)}{2}\right)^2 - C(v)} \quad (2.13)$$

gives the holomorphic embedding of  $\Sigma$  as a double cover over  $v$ -plane. In the regime of not too strong coupling, i.e. when  $|\Lambda|$  is small compared to the distances between the  $\phi_a$  and the  $m_i$ , we see that there is a pair of branch points, that we join by a branch cut, in the vicinity of each  $\phi_a$ . Note that the geometries of the two sheets are not identical. In particular, the upper sheet given by  $t_+(v)$  goes off to  $X^6 = +\infty$  (i.e.  $t_+ \rightarrow 0$ ) whenever  $C(v)$  vanishes, i.e. at  $v = m_i$ ,  $i = 1, \dots, N_f$ . See figure 1 below.



**Fig. 1:** The surface  $\Sigma$  as a double cover of the  $v$ -plane for the case of  $N_c = 5$  and  $N_f = 1$ . Five pairs of branch points (solid dots) are joined by branch cuts (wavy lines). The quark singularity on the upper sheet is represented by a hollow dot. We have also indicated representatives of three different homology cycles on  $\Sigma$ . Dashed and dotted contours are on the lower and upper sheet respectively. Note that the homology cycle denoted  $W$  is represented by two closed curves of opposite orientation on the upper sheet.

### 3. Supersymmetric two-branes

We will now consider the BPS-saturated excitations of the vacuum state defined by the background metric and five-brane configuration described in the previous subsections. These states are realized by supersymmetric configurations of two-branes. Since  $\Sigma$  plays the role of the spectral curve in the Seiberg-Witten formalism, the two-brane must have boundaries on  $\Sigma$  to constitute a charged state. Since  $\Sigma$  lies at a single point in  $\mathbf{R}^3$ , it is reasonable that this should hold for the two-brane as well. That is, we now consider a two-brane excitation, the world-volume of which spans a world-line in  $\mathbf{R}^{1,3}$ , lies at a single point in  $\mathbf{R}^3$  and defines a two-dimensional surface  $D$  in  $Q^4$ . The boundary  $C = \partial D$  of  $D$  lies on  $\Sigma$  so that the two-brane ends on the five-brane [9].

The general form of the lower-bound for the area of a brane found in the previous section equally applies for the two-brane. The area is thus again bounded from below by a topological quantity given as the integral of the pull-back to the two-brane surface  $D$  of the



Kähler-form corresponding to some complex structure  $J'$ . Suppose now that the bound is saturated, so that  $D$  is holomorphically embedded with respect to  $J'$ . The complex structure  $J'$  cannot equal the complex structure  $J$  with respect to which the surface  $\Sigma$  is holomorphically embedded, because this would mean that the intersection  $C$  of  $D$  and  $\Sigma$  would also be a holomorphically embedded submanifold of  $Q^4$  and thus in particular could not be a curve of real dimension one. An obvious guess is then to require that  $J'$  should be orthogonal to  $J$ . Given  $J$ , the set of such  $J'$  is parametrized by an  $S^1$ , that would naturally correspond to the phase of the central charge of the BPS-saturated state.

To see this more explicitly, let us construct the unbroken supersymmetry generators in the presence of the five-brane: We decompose an eleven-dimensional vector, such as the energy-momentum  $P_M$ ,  $M = 0, \dots, 10$  into a vector  $P_\mu$ ,  $\mu = 0, \dots, 3$  in  $\mathbf{R}^{1,3}$ , a vector  $P_m$ ,  $m = 7, 8, 9$  in  $\mathbf{R}^3$  and a vector  $P_s$ ,  $s = 4, 5, 6, 10$  in  $Q^4$ . An eleven-dimensional spinor, such as the supercharges  $Q_A$ ,  $A = 1, \dots, 32$  is written as  $Q_{\alpha ai}$ , where  $\alpha = 1, \dots, 4$ ,  $a = 1, 2$  and  $i = 1, \dots, 4$  index a spinor on  $\mathbf{R}^{1,3}$ ,  $\mathbf{R}^3$  and  $Q^4$  respectively. The supercharges that are left unbroken by the background metric and five-brane configuration described above can be written schematically as

$$\eta_{\alpha a} = \Pi_{\alpha\beta}^+ \zeta_i^+ Q_{\beta ai} + \Pi_{\alpha\beta}^- \epsilon_{ab} \zeta_i^- Q_{\beta bi}, \quad (3.1)$$

where  $\Pi^\pm$  are the projectors on positive and negative  $\mathbf{R}^{1,3}$  chirality,  $\epsilon_{ab}$  is anti-symmetric and  $\zeta^\pm$  are the covariantly constant spinors on  $Q^4$  that we discussed above. This particular expression is valid to the extent that the  $\zeta^\pm$  are uniform in the local coordinates chosen. More generally, one must actually include  $\zeta^\pm$  inside the spatial integration that defines the global supersymmetry generators. The  $\eta_{\alpha a}$  in (3.1) are constructed so that they are Majorana spinors on  $\mathbf{R}^{1,3}$ . To see this, recall that the supercharges  $Q_A$  form a Majorana spinor on  $\mathbf{R}^{1,10}$ . Furthermore, the charge conjugation matrix on  $\mathbf{R}^{1,10}$  is the tensor product of the charge conjugation matrices on  $\mathbf{R}^{1,3}$ ,  $\mathbf{R}^3$  and  $Q^4$ , the second of which equals the antisymmetric tensor  $\epsilon_{ab}$ . Finally, the covariantly constant spinors  $\zeta^+$  and  $\zeta^-$  are each others charge conjugates on  $Q^4$ .

Inserting the form of  $\eta_{\alpha a}$  in the 11-dimensional supersymmetry algebra (2.2), we find that they fulfill

$$\{\eta_{\alpha a}, \bar{\eta}_{\beta b}\} = (\gamma^\mu)_{\alpha\beta} P_\mu \delta_{ab} + \epsilon_{ab} \left( \Pi_{\alpha\beta}^+ Z + \Pi_{\alpha\beta}^- \bar{Z} \right). \quad (3.2)$$

Here we have normalized  $\zeta^+$  and  $\zeta^-$  by  $\bar{\zeta}^+ \zeta^+ = \bar{\zeta}^- \zeta^- = 1$ , and the central charge  $Z$  is given by

$$Z = \int_D \Omega_D, \quad (3.3)$$

where  $\Omega_D$  is the pull-back to  $D$  of the covariantly constant two-form  $\bar{\zeta}^- \gamma_{st} \zeta^+ dX^s \wedge dX^t$  on  $Q^4$ . As the notation suggests and as a little algebra shows, the latter two-form in fact equals the covariantly constant *holomorphic*  $(2, 0)$  form  $\Omega$  of  $Q^4$  that we described in the last section.

We can now use an identity analogous to (2.5):

$$\frac{1}{4} \left( (*K_D)^2 + |*\Omega_D|^2 \right) = 1, \quad (3.4)$$

where  $K_D$  is the pull-back of the Kähler form  $K$  to  $D$ , and the  $*$  denotes the Hodge-dual with respect to the induced metric on  $D$ . It follows that the area  $A_D$  of  $D$  fulfills the inequalities in

$$2A_D = 2 \int_D V_D = \int_D V_D \sqrt{(*K_D)^2 + |*\Omega_D|^2} \geq \int_D |*\Omega_D| V_D \geq \left| \int_D \Omega_D \right| = |Z|, \quad (3.5)$$

where  $V_D$  is the volume-form of  $D$ . The first inequality is saturated if and only if the pull-back  $K_D$  of the Kähler form vanishes, while the second is saturated if and only if the phase of the pull-back  $*\Omega_D$  of the holomorphic two-form is constant over  $D$ . This constant phase of  $*\Omega_D$  tells us that there is a second Kähler form whose pull-back to  $D$  vanishes. The surface  $D$  is then a holomorphic embedding with respect to some complex structure  $J'$  which is orthogonal to the complex structure  $J$ . Given  $J$ , there is an  $S^1$  of possible such  $J'$ , corresponding to the phase of  $*\Omega_D$ , i.e. the phase of the central charge  $Z$ . Different supersymmetric surfaces  $D$  will in general have different constant phases of  $*\Omega_D$ , and will be holomorphic with respect to different  $J'$ .

Assuming that  $\Omega_D$  is exact on  $D$ , i.e.  $\Omega_D = d\lambda$  for some one-form  $\lambda$ , the central charge can be rewritten as a boundary integral,

$$Z = \int_D \Omega_D = \oint_C \lambda. \quad (3.6)$$

This defines a one-form  $\lambda$  on  $\Sigma$  up to an exact holomorphic form. This one-form on  $\Sigma$  is closed because  $d\lambda$  is the pull-back of the holomorphic two-form  $\Omega$  to holomorphically embedded surface  $\Sigma$ , and must vanish identically. We have thus recovered the Seiberg-Witten expression for the central charge as the period of a certain meromorphic one-form  $\lambda$  on the spectral curve  $\Sigma$ .

### 3.1. The case of $N = 2$ QCD

Coming back to the example of  $SU(N_c)$  super Yang-Mills theory with  $N_f$  flavors, we can see that the holomorphic two-form  $\Omega = ds \wedge dv$  is indeed exact in this case. A possible choice for  $\lambda$  is

$$\lambda = -v ds_{\pm}(v) = v \frac{dt_{\pm}(v)}{t_{\pm}(v)}, \quad (3.7)$$

where  $t_{\pm}(v)$  solve the equation  $t^2 + B(v)t + C(v) = 0$ . This form of  $\lambda$  was also noted in [10][11]. More explicitly, one finds

$$\lambda = -v dv \frac{1}{2t_{\pm}(v) + B(v)} \left( B'(v) + \frac{C'(v)}{t_{\pm}(v)} \right). \quad (3.8)$$

Note that  $\lambda$  possesses a pole on the upper sheet when  $t_+(v) = 0$ , which happens whenever  $C(v) = 0$ , i.e., at  $v = m_i$ . This pole is associated with the bare mass of a quark and has residue equal to  $m_i$ .

In this example, it is also easy to construct the complex structures that are orthogonal to the one with respect to which  $s$  and  $v$  are holomorphic. Such a complex structure is specified by an angle  $\theta$  and can be described by stating that the variables

$$\begin{aligned} z &= \frac{1}{2} (s + \bar{s} + e^{i\theta}v - e^{-i\theta}\bar{v}) \\ w &= \frac{1}{2} (-s + \bar{s} + e^{i\theta}v + e^{-i\theta}\bar{v}) \end{aligned} \quad (3.9)$$

are holomorphic. In these variables, the original Kähler form and the holomorphic two-form (2.8) and (2.9) are

$$\begin{aligned} K &= i(dz \wedge dw - d\bar{z} \wedge d\bar{w}) \\ \Omega &= \frac{1}{2} e^{-i\theta} (-dz \wedge d\bar{z} - dw \wedge d\bar{w} + dz \wedge dw + d\bar{z} \wedge d\bar{w}), \end{aligned} \quad (3.10)$$

so  $K$  indeed vanishes and  $\Omega$  has a constant phase determined by  $\theta$  when pulled back to a surface  $D$  which is holomorphically embedded with respect to this complex structure.

## 4. The BPS-saturated spectrum

In this section, we will discuss the spectrum of BPS-saturated states in general  $N = 2$  theories. Although Seiberg-Witten theory gives a way of calculating the central charge, and thus the mass, of a BPS-saturated state with given quantum numbers, it does not

address the question of whether such a state really exists in the spectrum. The  $M$ -theory approach in principle provides an answer; the existence of a state amounts to the existence of a supersymmetric surface  $D$ , the boundary of which lies on the spectral curve  $\Sigma$  and represents a specific homology class given by the quantum numbers of the state. As we have seen, such a surface is holomorphically embedded with respect to a complex structure that is given by the phase of the central charge.

It is important to emphasize that the BPS condition requires more than the simple minimization of the mass. Given a homology class of the boundary, there will in general exist surface of minimal area, but there is no guarantee that it actually saturates the BPS bound. Conversely, a generic supersymmetric surface  $D$ , i.e. a surface that is holomorphically embedded with respect to the complex structure  $J'$ , will only intersect the surface  $\Sigma$  at isolated points and thus fail to be a finite surface. Furthermore, even if  $D$  intersects  $\Sigma$  along a closed curve  $C$ , it may still go off to infinity in  $Q^4$  somewhere in the interior. And here lies the difficulty in finding the actual BPS spectrum.

#### 4.1. The case of $N = 2$ QCD

Before proceeding with the more general discussion, we would like to present some examples of exact supersymmetric two-brane configurations in the case of  $N = 2$  QCD that we have been discussing. We can describe the quantum numbers of a state by giving the homology class of the boundary on  $\Sigma$  of the corresponding surface  $D$ . At not too strong coupling, the BPS-saturated spectrum is known to consist of the following states:

1. Electrically charged vector mesons  $W$  in vectormultiplets.
2. Magnetically charged monopoles  $M$  in hypermultiplets.
3. Electrically charged quarks  $Q$ , that also carry a quark number charge, in hypermultiplets.
4. Electrically and magnetically charged dyons  $D$  in hypermultiplets.
5. Quark-soliton bound states  $B$  in hypermultiplets.

Representatives of the homology classes corresponding to  $W$ ,  $M$  and  $Q$  were indicated in figure 1 above. Some of these states may decay as they cross a domain wall of marginal stability.

We will now give examples of the  $D$ -surfaces corresponding to vectormultiplets, monopoles and quarks. We will do this in a limit where the branch points and quark

singularities not relevant for the state in question are far away and thus can be disregarded. To be able to give exact solutions, we will also impose a certain reality condition on the moduli.

For the vector mesons and the monopoles, the condition that the irrelevant features of  $\Sigma$  are far away means that we are effectively considering the pure  $SU(2)$  theory, i.e. the surface is given by the simple equation

$$t = t_{\pm}(v) = -\frac{v^2 - \phi^2}{2} \pm \sqrt{\left(\frac{v^2 - \phi^2}{2}\right)^2 - \frac{1}{4}}, \quad (4.1)$$

where  $\phi = \phi_1 = -\phi_2$  is the complex adjoint Higgs expectation value and we have set the scale  $\Lambda = 1/\sqrt{2}$ . The square-root branch cuts emanate from the points  $v = \pm\sqrt{\phi^2 + 1}$  and end at the points  $v = \pm\sqrt{\phi^2 - 1}$ .

The reality condition that we will impose on the modulus in order to be able to construct exact BPS-saturated two-branes amounts to taking  $\phi$  to be real and  $|\phi| > 1$  so that all the branch points lie on the real  $v$ -axis. (One could also take  $\phi$  to be imaginary and  $|\phi| > 1$ , in which case all branch points would lie on the imaginary  $v$ -axis.)

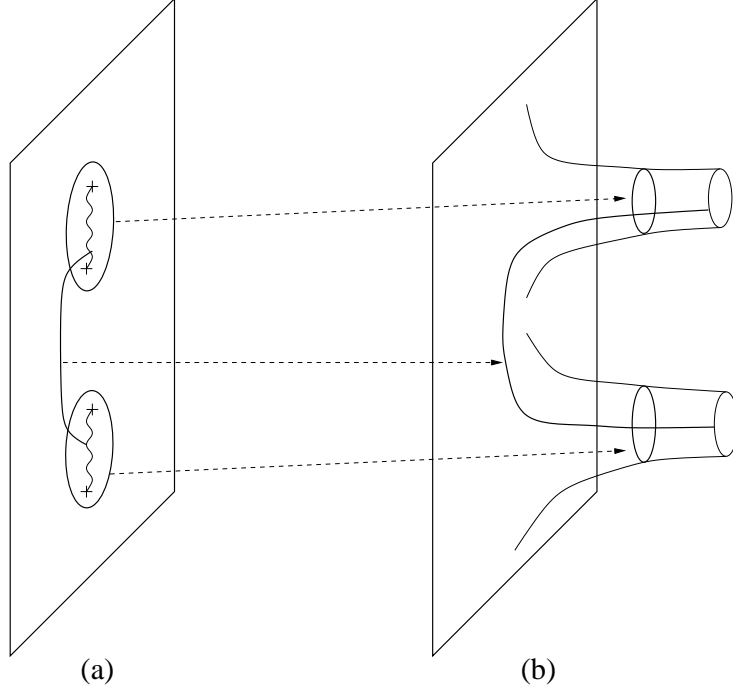
In the vicinity of one of the branchpoints, say at  $v = v_0$ , the induced metric on  $\Sigma$  has the form

$$g_{\Sigma} = \left| \frac{dt_{\pm}(v)}{t_{\pm}(v)} \right|^2 + |dv|^2 \simeq |v_0| \left| \frac{dv}{\sqrt{v - v_0}} \right|^2, \quad (4.2)$$

which means that  $v - v_0$  covers only half of the local neighborhood. A better holomorphic coordinate would be  $y = \sqrt{v - v_0}$ , for which one finds

$$g_{\Sigma} \simeq 4|v_0||dy|^2. \quad (4.3)$$

For instance, a closed path on  $v$ -plane that tightly winds around a pair of such singular points  $\sqrt{\phi^2 + 1}$  and  $\sqrt{\phi^2 - 1}$  is actually a smooth circle on the surface  $\Sigma$ , although it more looks like a pair of straight segments between the two branch points in the  $v$ -plane. See figure 2 below.



**Fig. 2:** A half of the surface  $\Sigma$ . The first figure shows the square root cuts (wiggly lines) in the  $v$ -plane, while the second illustrates the actual shape of the surface. The second half given by the other sheet is smoothly attached to the two circles to the right. As one tightens the closed paths around the branch cuts in (a), the corresponding circles in (b) move toward the other half of  $\Sigma$ .

For the vector meson  $W$ , one now considers the surface

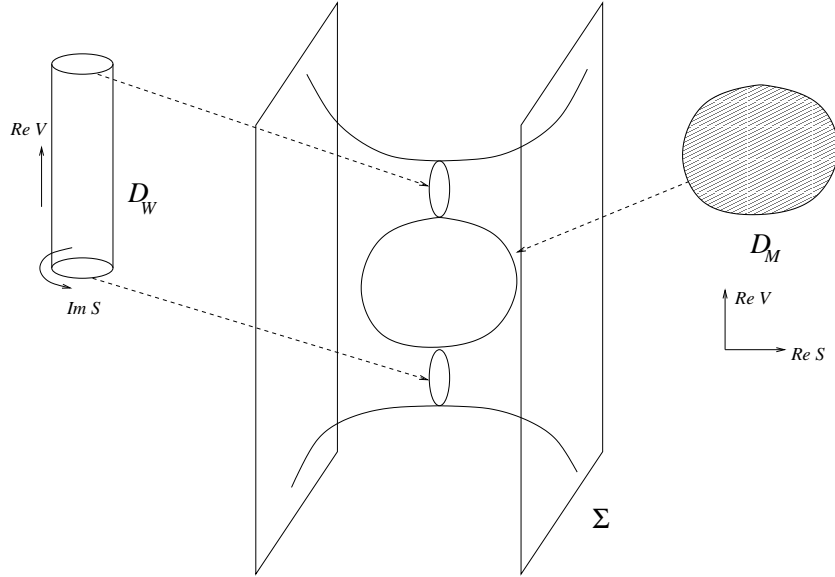
$$|t| = \frac{1}{2}, \quad v = \bar{v}. \quad (4.4)$$

The pull-back of the Kähler form clearly vanishes, and it is also easy to see that the pull-back of  $\Omega$  is purely imaginary. It thus saturates the conditions to be a supersymmetric surface. Can we also make it into a bounded surface? In fact, along the straight line segment on the  $v$ -plane between  $v = \sqrt{\phi^2 + 1}$  and  $v = \sqrt{\phi^2 - 1}$ , and also along the straight line segment between  $v = -\sqrt{\phi^2 + 1}$  and  $v = -\sqrt{\phi^2 - 1}$ , we have  $|t| = \frac{1}{2}$  on  $\Sigma$ . As noted above, these are really closed curves on  $\Sigma$  winding around the two tubes that connect the upper and lower sheets. Cutting off the surface (4.4) along these curves defines the vectormultiplet surface  $D_W$ , which thus has the topology of a cylinder. An important fact is that the two boundary curves are of opposite orientations on the upper sheet of  $\Sigma$ .

For the monopole  $M$ , we take the surface<sup>†</sup>

$$t = \bar{t}, \quad v = \bar{v}. \quad (4.5)$$

Again the BPS condition is satisfied trivially, and we only need to ask if this surface contains a bounded component. In fact, both the upper and the lower sheets of  $\Sigma$  intersect with this surface along the straight line segments on the  $v$ -plane between  $v = -\sqrt{\phi^2 - 1}$  and  $v = \sqrt{\phi^2 + 1}$ . Cutting off the surface (4.5) along the closed curve formed by these segments defines the monopole surface  $D_M$ , which thus has the topology of a disc. Note that  $D_M$  actually collapses to a point as  $\phi \rightarrow 1$ . Of course, this simply reproduces the well-known strong-coupling singularity on the moduli space of vacua where the monopole becomes massless. The surfaces  $D_W$  and  $D_M$  are depicted in figure 3 below.



**Fig. 3:** The BPS-saturated surfaces  $D_W$  and  $D_M$  with boundaries on  $\Sigma$  corresponding to the vector meson and the monopole.

For the quark  $Q$ , we consider a situation where some  $\phi_a$  is close to some  $m_i$  and the others are far away. The reality condition will be that we are at weak coupling, i.e. that  $\text{Re } \log \Lambda \rightarrow -\infty$ . For simplicity, we shift and rescale  $v$  so that  $\phi_a = 1$  and  $m_i = -1$ . The upper sheet of  $\Sigma$  is then given by

$$s = \log \frac{v+1}{v-1}, \quad (4.6)$$

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<sup>†</sup> If one is considering the case of purely imaginary  $\phi$ , the same kind of construction gives the dyon instead.

where we have shifted  $s$  by  $\log \Lambda$ . The quark surface  $D_Q$  should be holomorphically embedded with respect to the complex structure in which

$$\begin{aligned} z &= \frac{1}{2}(s + \bar{s} + v - \bar{v}) \\ w &= \frac{1}{2}(v + \bar{v} - s + \bar{s}) \end{aligned} \tag{4.7}$$

are holomorphic variables. We note that  $\Sigma$  is invariant under the transformations  $(s, v) \rightarrow (\bar{s}, \bar{v})$  and  $(s, v) \rightarrow (-s, -v)$ . By symmetry, the same should hold for the quark surface  $D_Q$ , which must therefore be given by an equation of the form

$$w = f(z), \tag{4.8}$$

where the holomorphic function  $f$  is real for real argument and also odd, i.e.  $f(-z) = -f(z)$ . It follows that  $D_Q$  cannot intersect the upper sheet of  $\Sigma$  anywhere on the straight line segment on the  $v$ -plane between  $v = -1$  and  $v = 1$ , since here we have  $z$  real but  $\text{Im } w = \pm\pi$ . Although we do not know the exact form of  $f(z)$ , it is not difficult to numerically find a holomorphically embedded surface that comes close to having a boundary on  $\Sigma$  and thus approximates the true surface  $D_Q$ . To this end, we note that  $f(z)$  can be expanded in a power series as

$$f(z) = \sum_{k=0}^{\infty} c_{2k+1} z^{2k+1}, \tag{4.9}$$

where the coefficients  $c_{2k+1}$  are real. To have a finite area solution for  $D_Q$ , this series must converge everywhere on  $D_Q$ . We can then approximate  $f(z)$  by truncating the series after the first few terms. In this way, one finds that the surface  $D_Q$  is an almost flat disc bounded by an ellipse-like curve on the upper sheet of  $\Sigma$  that surrounds, but does not approach, the singular points at  $v = -1$  and  $v = 1$ .

#### 4.2. The topology of vectormultiplets and hypermultiplets

We have discussed a few specific examples of supersymmetric two-branes as  $N = 2$  BPS-saturated states. An interesting feature is that the vector meson, which is a vector-multiplet, is realized as a cylinder with two boundaries, whereas the monopole and quark,



which are hypermultiplets, are realized as discs with a single boundary<sup>\*</sup>. In fact, the homology classes on the spectral curve that correspond to vector mesons are represented by closed contours in opposite directions around two pairs of branch points. This has to be the boundary of a cylinder or two separate discs. However, the latter possibility would mean that each disc existed separately as a BPS-saturated state, which is known not to be the case. Also, all other known cases of BPS-saturated states, i.e. solitons and quarks, have homology classes that can be represented by a single closed contour that could be the boundary of a disc. These examples lead us to formulate a conjecture: Surfaces with the topology of a cylinder and a disc correspond to, respectively, BPS-saturated vector-multiplets and hypermultiplets in the  $N = 2$  and  $D = 4$  theories. (Given the severe constraints on supersymmetric surfaces with boundary on  $\Sigma$ , it seems unlikely that such a surface could have more than two boundary components, except at isolated points in the moduli space of  $\Sigma$ .) It would be interesting to confirm this conjecture directly from the world-volume theory of the two-brane and the five-brane.

#### 4.3. *The phenomenon of marginal stability*

By charge and energy conservation, BPS-saturated states are stable at a generic point in the moduli space of vacua. Furthermore, the spectrum of such states is in general stable under small variations of the moduli. The only exception is when the quantum numbers of a BPS-saturate state is the sum of the quantum numbers of two other BPS-saturated states, and the central charges of the three states have the same phase. As such a domain wall of marginal stability in the moduli space of vacua is approached, the mass of the heaviest state approaches the sum of the masses of the lighter states. It is then *possible* for it to decay exactly at the domain wall and be absent from the spectrum on the other side. We emphasize that these conditions are necessary but not sufficient for the decay of a BPS-state.

The  $M$ -theory picture gives some further insight on this issue. The stability of BPS-states away from their domain walls of marginal stability means that given the two surfaces  $\Sigma$  and  $D$ , it must in general be possible to accommodate a small deformation of  $\Sigma$  by a

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<sup>\*</sup> We can also realize the matter sector by including six-branes in a type IIA description or taking  $Q^4$  to be a Taub-NUT space in the  $M$ -theory description [1]. The quark is then represented as a string stretching from a four-brane to a six-brane or as a surface with a boundary on  $\Sigma$  that winds around the compact direction of  $Q^4$ . The other end of the surface is at a Taub-NUT center, where the compact direction degenerates to a point, so this is a disc rather than a cylinder.

deformation of  $D$  so that  $D$  still has its boundary on  $\Sigma$ . (Obviously we only consider deformations that preserve the property that  $\Sigma$  and  $D$  are holomorphically embedded with respect to orthogonal complex structures.)

For a hypermultiplet to decay into two other hypermultiplets, the corresponding disc surface must degenerate into two discs whose boundaries touch in a point. This means that the intersection number of the corresponding homology classes on  $\Sigma$  is  $\pm 1$ , or, equivalently, that the symplectic product of the electric-magnetic charge vectors equals  $\pm 1$ . These states are thus mutually non-local. As an example, we take a dyonic soliton and a quark in  $SU(2)$  Yang-Mills theory with a fundamental flavor. The corresponding electric-magnetic charges  $(q_e, q_m)$  and  $(q'_e, q'_m)$  are  $(2n, 1)$  and  $(1, 0)$  respectively, so that the symplectic product  $q_m q'_e - q_e q'_m$  equals 1. In a certain domain in the moduli space, these states can form a bound state [12] that decays into its constituents as the corresponding domain wall of marginal stability is crossed. More general situations of this kind were studied in [13], with the result that decay of hypermultiplet bound states is indeed related to the mutual non-locality of the constituents.

Similarly, for a vectormultiplet to decay into two hypermultiplets, the corresponding cylinder surface must degenerate into two discs whose boundaries touch in two points. This means that the intersection number of the corresponding homology classes on  $\Sigma$  is  $\pm 2$ , so non-locality of the constituents is again an essential requirement. An example of this process can be found in pure  $SU(2)$  gauge theory, where the vector meson of electric-magnetic charge  $(2, 0)$  can decay into two dyons with charges  $(2n, 1)$  and  $(2 - 2n, -1)$  so that the symplectic product equals 2.

The decay processes described in the last two paragraphs are fairly easy to visualize. From a topological point of view, it seems much less natural for a state to decay into two vectormultiplets or a vectormultiplet and a hypermultiplet, at least at a generic point along a domain wall of marginal stability. We would thus like to end with the speculation that the only marginal stability decays that are possible are those that we have described, namely that either a hypermultiplet or a vectormultiplet decays into two hypermultiplets. A proof of this statement or a counterexample would be interesting.

We would like to thank Chalmers University of Technology (M.H.) and the Theory Division at CERN and the Aspen Center for Physics (P.Y.) for their hospitality. P.Y. is supported in part by U.S. Department of Energy.

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